Problem 1.2

(a) Find the standard deviation of the distribution in Example 1.2.

(b) What is the probability that a photograph, selected at random, would show a distance x more than one standard deviation away from the average?

Solution

The probability distribution in Example 1.2 is given by

$$\rho(x) = \frac{1}{2\sqrt{hx}}, \quad 0 \leq x \leq h.$$

Part (a)

To determine the standard deviation, use equation 1.19 in the textbook.

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \tag{1.19}$$

The aim then is to calculate $\langle x^2 \rangle$ and $\langle x \rangle^2$.

$$\langle x \rangle = \frac{\int_0^h x\rho(x)\,dx}{\int_0^h \rho(x)\,dx} = \frac{\int_0^h \frac{1}{2\sqrt{h}} x^{1/2}\,dx}{\int_0^h \frac{1}{2\sqrt{h}} x^{-1/2}\,dx} = \frac{\frac{1}{2\sqrt{h}} \cdot \frac{2h^{3/2}}{3}}{\frac{1}{2\sqrt{h}} \cdot 2h^{1/2}} = \frac{h}{3}$$

$$\langle x^2 \rangle = \frac{\int_0^h x^2\rho(x)\,dx}{\int_0^h \rho(x)\,dx} = \frac{\int_0^h \frac{1}{2\sqrt{h}} x^{3/2}\,dx}{\int_0^h \frac{1}{2\sqrt{h}} x^{-1/2}\,dx} = \frac{\frac{1}{2\sqrt{h}} \cdot \frac{2h^{5/2}}{5}}{\frac{1}{2\sqrt{h}} \cdot 2h^{1/2}} = \frac{h^2}{5}$$

Substituting these formulas into equation 1.19 gives

$$\sigma = \sqrt{\frac{h^2}{5} - \left(\frac{h}{3}\right)^2} = \sqrt{\frac{4h^2}{45}} = \frac{2h}{3\sqrt{5}} = \frac{2h\sqrt{5}}{15} \approx 0.298h$$

for the standard deviation.

Part (b)

Here we have to determine the probability of finding the rock more than one standard deviation from average. Do this by integrating the probability distribution over the appropriate interval(s) of x within $0 \leq x \leq h$.

$$\begin{aligned}
P &= \int_0^{\langle x\rangle-\sigma} \rho(x)\,dx + \int_{\langle x\rangle+\sigma}^{h} \rho(x)\,dx \\
&= \int_0^{\frac{h}{3}-\frac{2h\sqrt{5}}{15}} \frac{1}{2\sqrt{h}} x^{-1/2}\,dx + \int_{\frac{h}{3}+\frac{2h\sqrt{5}}{15}}^{h} \frac{1}{2\sqrt{h}} x^{-1/2}\,dx \\
&= \frac{1}{2\sqrt{h}} \cdot 2x^{1/2}\bigg|_0^{\frac{h}{3}-\frac{2h\sqrt{5}}{15}} + \frac{1}{2\sqrt{h}} \cdot 2x^{1/2}\bigg|_{\frac{h}{3}+\frac{2h\sqrt{5}}{15}}^{h} \\
&= \frac{1}{\sqrt{h}}\left(\sqrt{\frac{h}{3}-\frac{2h\sqrt{5}}{15}} - 0\right) + \frac{1}{\sqrt{h}}\left(\sqrt{h} - \sqrt{\frac{h}{3}+\frac{2h\sqrt{5}}{15}}\right) \\
&= \sqrt{\frac{1}{3}-\frac{2\sqrt{5}}{15}} + 1 - \sqrt{\frac{1}{3}+\frac{2\sqrt{5}}{15}} \\
&\approx 0.393
\end{aligned}$$